Analytical Forms for Internal Ballistics

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Abstract

The internal ballistics (barrel pressure and bullet velocity) are of interest in small firearms. There are modeling techniques to approximate these quantities. One of the earliest attempts was to model their characteristics using a polynomial [1, 2]. An improvement on this representation has been made. This representation allows the determination of barrel pressure and bullet velocity using the physical parameters of the small firearm, bullet and powder. Comparison of this solution is compared to that provided by a LEM model.

Keywords: internal ballistics, LEM, polynomial modeling

1. Introduction

There are numerous papers and books on internal ballistics. A number of expressions exist to model the barrel pressure and bullet velocity distributions before the bullet exits the barrel. Such expressions can be found in Corner [3], Coppock [4], Carlucci [5] and Miner [6]. The expressions provide an informative functional relationship relating gun, powder and bullet parameters for velocity and pressure distributions. Naturally, their accuracy may be limited considering the approximations required to obtain analytical, closed form representations for a complex process.

The work presented here extends the work of Challeat/Leduc[1] and Kolbe [2]. These works represent internal pressure and velocity quantities with polynomial expansions of order one. The polynomial expansions are used here but allow for a non-integer order exponent greatly improving the modeling accuracy. These expansions are then used in numerical techniques to arrive at the velocity and pressure distributions for a gun. One of the techniques provided generates the desired distributions using only parameters of the barrel length, bore diameter, powder weight, bullet weight, muzzle velocity and powder impetus.

2. Lumped Element Model

There are many numerical models available to calculate the internal ballistics quantities of pressure and velocity [7]. A model is required to demonstrate the validity of the polynomial representation presented here. Any model could be used for this purpose but simple model was chosen for this application. The work of Cronemberger [8] and Akcay [9] was used to generate a reference solution for this work. This solution is often called a lumped element model (LEM) and consists of a system of differential-algebraic equations. As outlined by [8], the equations are

$$\frac{dz}{dt} = \frac{\beta}{D_w} P^a \tag{1}$$

$$\frac{dx}{dt} = v \tag{2}$$

$$\frac{dv}{dt} = \frac{\pi r^2}{m_b} \left[P - P_{atm} - P_r\right] \tag{3}$$

with the P in Eq. 3 is given by

$$P = \frac{(\gamma - 1)(Q - W - E_{lost})}{V_g - m_g c} \tag{4}$$

and

$$Q = \frac{m_g F}{\gamma - 1.} + E_i \tag{5}$$

$$W = .5 m_b v^2 \tag{6}$$

$$E_{lost} = .26 \frac{m_g F}{\gamma - 1.} \tag{7}$$

$$V_g = V_c + (z - 1)\frac{m_p}{\rho} + \pi r^2 x \tag{8}$$

where the unknowns are v, x and z which are the velocity of the bullet, the position of the bullet and the normalized amount of burnt powder. The other parameters are described in Table 1. Equations 1-3, are solved by a 4th order Runge-Kutta solver. Using the parameters in Table 1, the internal ballistic solution for the unknowns are shown in Figures 1 through 3. This example used 43.7 grains of IMR 4350 powder with a Rem 7-08 case with a 162 grain bullet. The values for β, F and a were numerically optimized with the system of equations for a barrel length (barrel length + length of bullet engraving) of 20.6 inches and a muzzle velocity of 2550 ft/s. The optimization algorithm used was from the Python scipy.optimize.minimize package. Note the discontinuity in the calculated pressure when all the powder is burnt suggests a model limitation. However, this model has matched peak pressure and location for other tested loads from chamber pressure measurements. Because of this, it is felt that the LEM model provides an adequate comparison reference for other analytical models presented here.

Table 1: Lumped method input data

Parameter	Value
Projectile radius, r	.0036068 m
Projectile mass, m_b	162 grns
Propellant mass, m_p	43.7 grns
Propellant density, ρ	1523 kg/m^3
Web thickness, D_w	4.4 10^{-4} m
Propellant burning rate constant, β	8.1637 10^{-7}m/s/Pa^{-a}
Propellant pressure index, a	.7009
Propellant impetus, F	1024.56 10^3 J/kg
Specific heat ratio, γ	1.22
Form function coefficient, k	0
Co-volume, c	.00089 m/kg
Primer energy, E_i	69 J
Pressure, resistance, P_r	5.9 10^6 Pa
Pressure, atmospheric, P_a	1.0 10^5 Pa
Case volume, V_c	3.1844 10^{-6} m^3
Barrel length, L	.5235 m

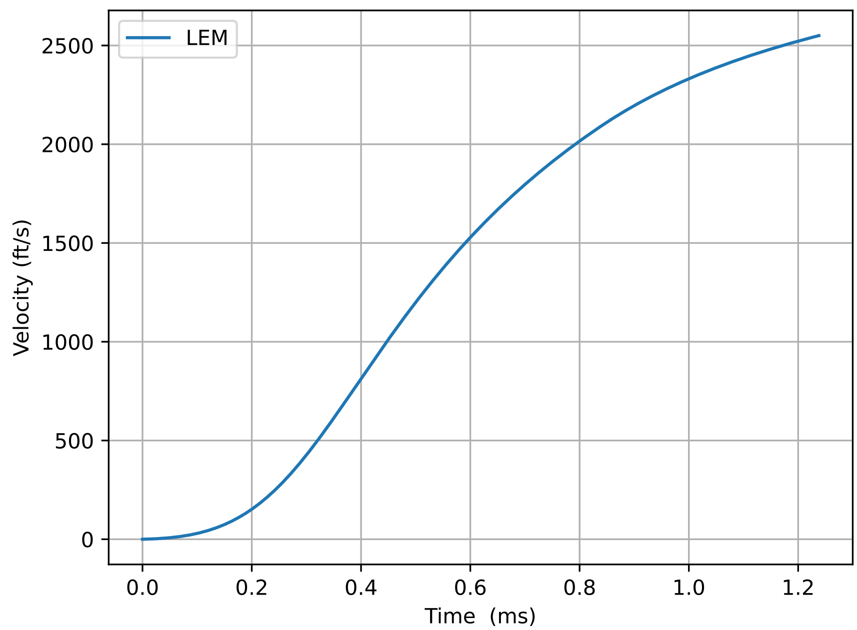

Figure 1: Calculated bullet velocity for the bullet in the barrel for LEM.

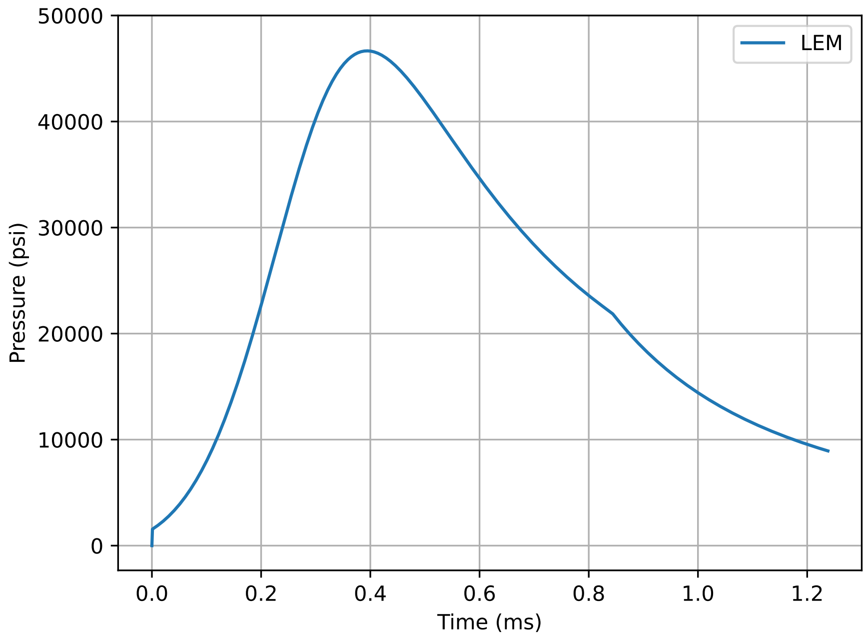

Figure 2: Calculated pressure on the bullet for the bullet in the barrel for LEM.

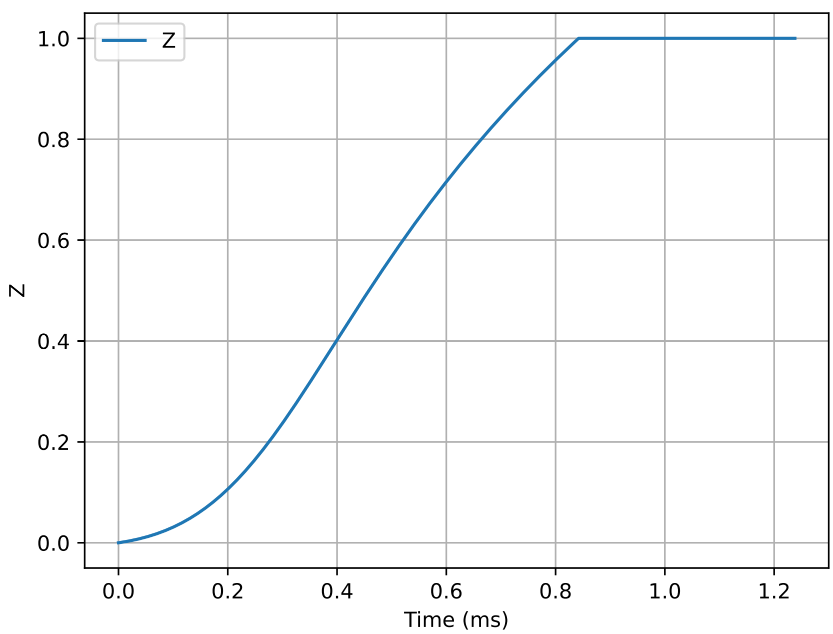

Figure 3: Calculated normalized burnt powder for LEM.

3. Polynomial Representation

Eqs. 9 and 10 form the basis of this work. The difference between the work of [1] and [2] and what is done here is that the spatial factors have exponents, n, other than unity. Allowing the spatial factors being raised to a rational power permits excellent agreement with the lumped element model. The polynomial model is given as

$$\text{velocity} = v = \frac{ax^n}{(b+x)^n} \tag{9}$$

$$\text{pressure} = \frac{m_b \cdot acc}{A} = \frac{m_b}{A}\frac{a^2bx^{(2n-1)}}{(b+x)^{(2n+1)}} \tag{10}$$

where acc is the bullet acceleration given by

$$\text{acceleration} = acc = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v \tag{11}$$

where

$$acc = \frac{a^2bx^{(2n-1)}}{(b+x)^{(2n+1)}} \tag{12}$$

using

$$\begin{aligned}
\frac{dv}{dx} &= \frac{anx^{(n-1)}(b+x)^n - anx^n(b+x)^{n-1}}{(b+x)^{2n}} \\
&= \frac{anx^n(b+x)^n\left(x^{-1} - (b+x)^{-1}\right)}{(b+x)^{2n}} \\
&= \frac{anx^n(\frac{b}{x(b+x)})}{(b+x)^n} \\
&= \frac{abnx^{(n-1)}}{(b+x)^{n+1}}
\end{aligned}$$

where a and b are coefficients, x is the bullet position, A is the cross-sectional area of the bore and m is the bullet mass. The dimensions of a is that of velocity and b is that of distance.

Excellent agreement occurs between the LEM and polynomial model (Eqs. 9 and 10). The polynomial model unknowns were found through a curve fitting procedure using the lumped element model data as shown in the following Figures 4 and 5. The Python curve fitting package scipy.optimize was used to determine the unknowns.

Curve fitting the velocity expression with the numerical solution yields coefficient values of $a = 3150$ ft/s, $b = .6233$ ft $= 7.48$ in and $n = .6879$. Figure 4 illustrates this comparison.

Curve fitting the pressure expression with the numerical solution yields coefficient values of $a = 3175$ ft/s, $b = ..5816$ ft $= 6.98$ in and $n = .7089$ with

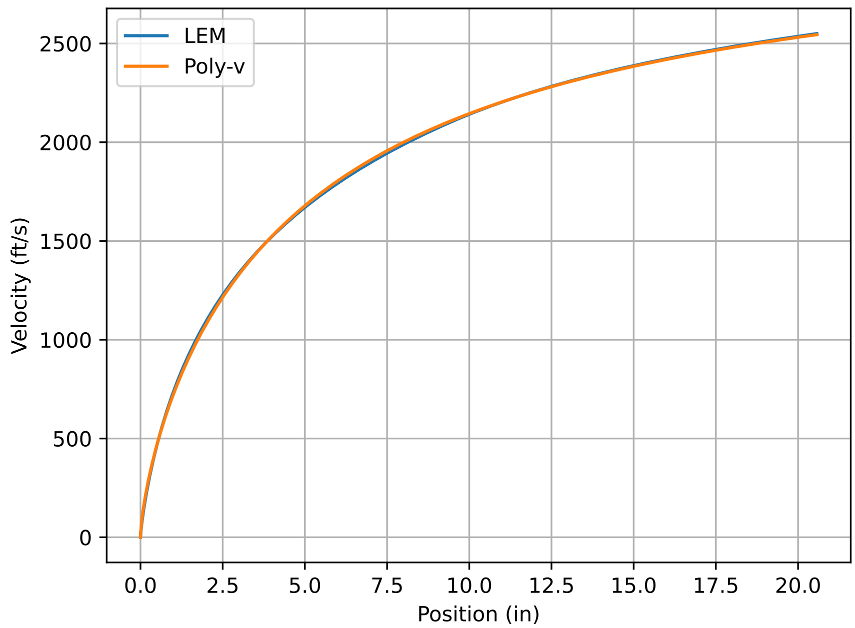

Figure 4: Illustration of velocity solutions for the LEM and polynomial models.

$m = 162$ grains $= .0007193$ slugs and $A = .06334$ sq. in. Figure 5 illustrates this comparison.

Naturally, the two sets of parameters, (one set from the velocity based curve fitting and the other from the pressure based curve fitting) should be the same. The coefficients for these cases are given in the following table.

Table 2: Curve fitted coefficients

Parameter	Velocity based	Pressure based
a (ft/s)	3150	3175
b (in)	7.48	6.98
n	.6879	.7089

The LEM also provides a numerical relationship between the position of the bullet as a function of time. A corresponding relationship is possible with the polynomial model. From Eq. 9 ($v = \frac{dx}{dt}$), time as a function of x can be described as

$$\int_0^{t_0} dt = \int_0^{x_0} \frac{(b+x)^n}{ax^n} dx \tag{13}$$

where x_0 and t_0 are the location and time values for the current bullet location. Unfortunately, this integral has no general, simple analytical result. However, this integral can be simply numerically integrated. Figure 6 illustrates the LEM and polynomial models for the temporal location of the

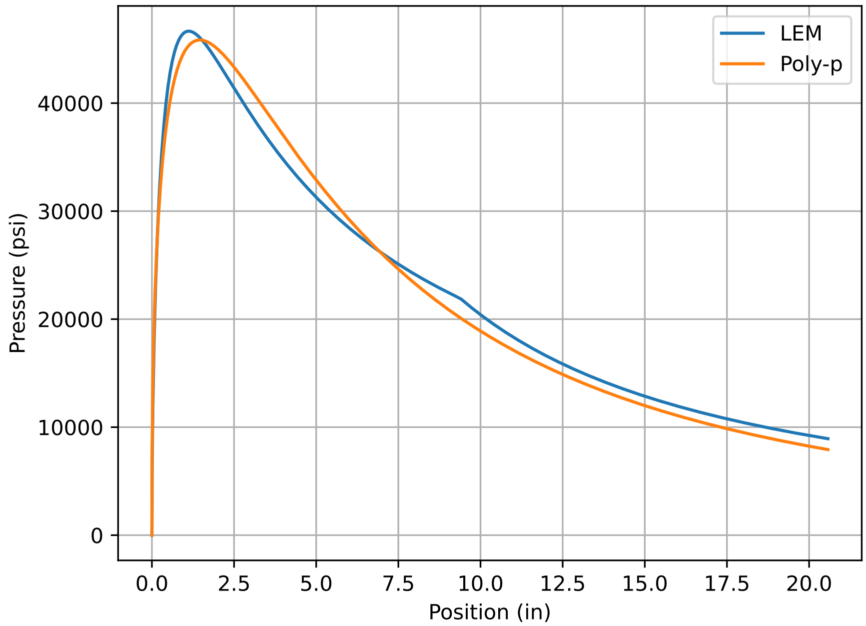

Figure 5: Illustration of pressure solutions for the LEM and polynomial models.

bullet for the current example. Excellent agreement exists between these two solutions which is reasonable considering the velocity comparison is good as well.

4. Coefficient Determination

The a and b parameters can be determined from the numerical solution as described in [2]. The velocity represented by the a parameter is the muzzle velocity with an infinitely long barrel ($x- > \infty$ (Eq. 9). The b parameter can be determined from the location of the peak pressure value which is obtained from setting the spatial derivative of Eq. 10 to zero. This results in

$$b = \frac{1}{n - .5} x_{p_m} \tag{14}$$

Inserting this x location in Eq. 9 yields the velocity at peak pressure, given by

$$v_{p_m} = a \frac{(n - .5)^n}{(n + .5)^n} \tag{15}$$

Kolbe [2] also provided a way to relate the gun, powder and bullet parameters to the polynomial coefficients for polynomials with unity powers. An alternate approach is provided here using the lump element model.

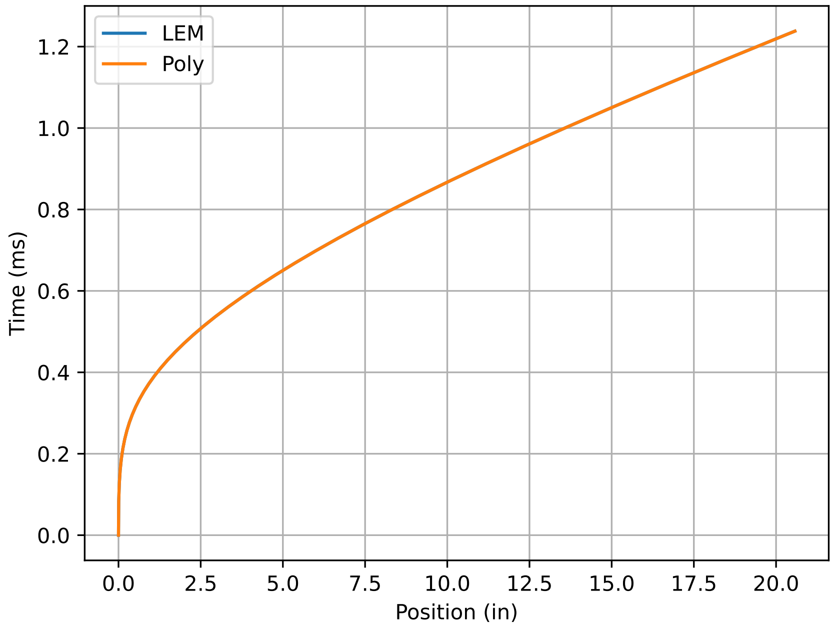

Figure 6: Illustration of time vs bullet location solutions for the LEM and polynomial models.

Expanding Eq. 3 for when $z = 1$ (powder has all burnt) yields the following functional

$$\text{functional} = abs(P - P_b - P_{atm} - P_r) \tag{16}$$

where

$$P = \frac{(1. - .26)Fm_p + (\gamma - 1.)(E_i - m_b v^2/2.)}{V_g - m_p * c} \tag{17}$$

and

$$P_b = \frac{m_b}{A}\frac{dv}{dt} \tag{18}$$

with

$$V_g = V_c + Ax. \tag{19}$$

The only unknowns in Eq. 16 are v and $\frac{dv}{dt}$ which can be expressed as polynomials using Eqs. 9 and 11. The factor A is the cross-sectional area of the bullet.

Using this functional, Eq. 16 with the Python scipy.optimize.minimize package, the coefficients b and n can be found. The a coefficient value is provided by

$$a = v_m * (b + x_m)^n / x_m^n \tag{20}$$

where v_m is the muzzle velocity and x_m is the travel distance of the bullet through the muzzle. The values for v_m and x_m are used in Eqs. 17 and 19 for v and x, respectively.

The Table 3 contains the coefficients found and compares them to the coefficients found earlier.

Table 3: Curve fitted coefficients

Parameter	Velocity based	Pressure based	Functional based
a (ft/s)	3150	3175	3154
b (in)	7.48	6.98	6.26
n	.6879	.7089	.8017

Figures 7 and 8 compare the polynomial solution using the coefficients found from the functional minimization to the lumped element solution. The agreement is not as good as with the comparisons in Figure 4 and 5 but not unreasonable. Note that the functional is not dependent upon the burning coefficient β or the pressure index exponent a but still yields a reasonable solution for the velocity and pressure distributions.

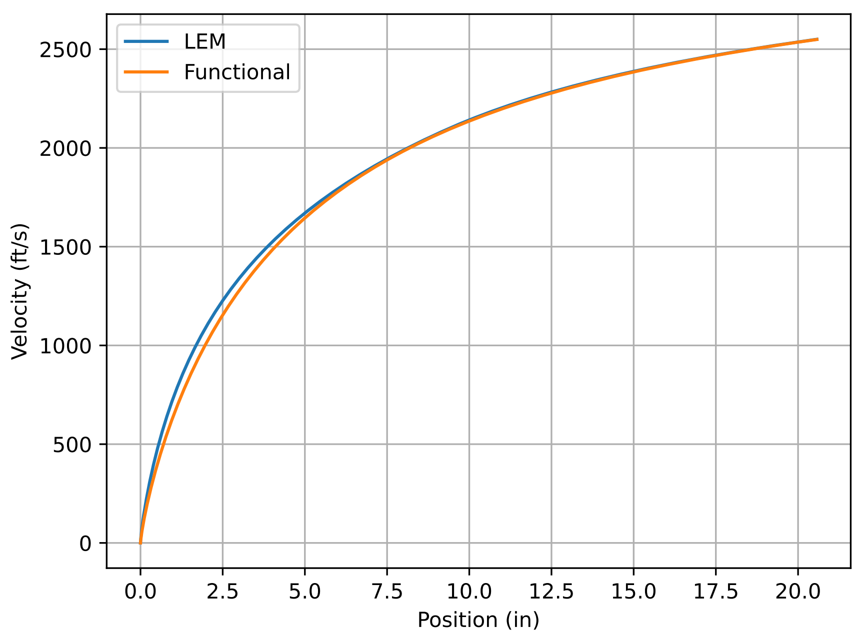

Figure 7: Illustration of velocity solutions for the LEM and functional models.

Note that the both the velocity and pressure curves from the fitted coefficients using the functional does not match as well in the 0 to 5 inch region. This is not surprising since only muzzle velocity data was used but the major velocity and pressure trends between the LEM and polynomial result are

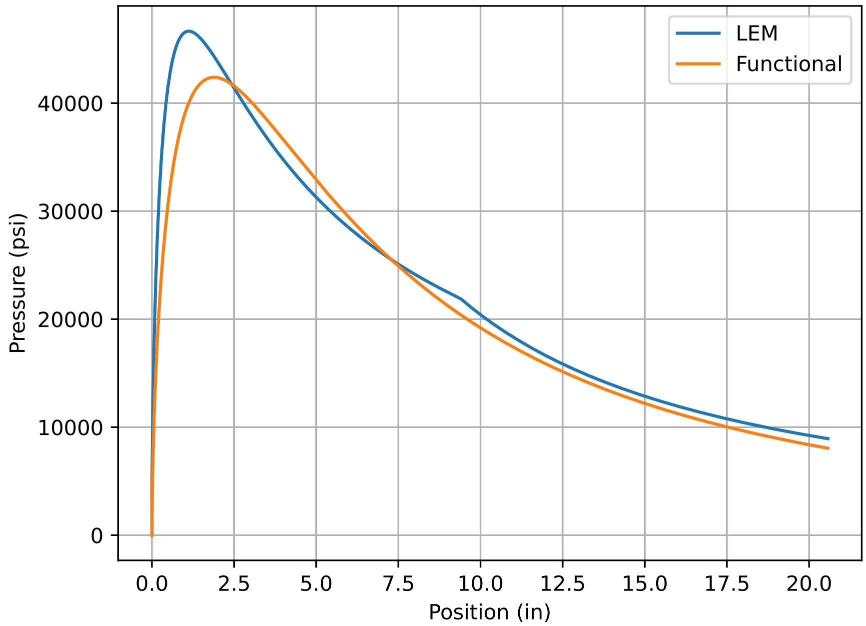

Figure 8: Illustration of pressure solutions for the LEM and functional models.

reasonable. Another polynomial velocity representation might yield better results for both the velocity and pressure curves (the pressure representation is derived from the derivative of the velocity representation.

5. Conclusion

An improved polynomial model for the velocity and pressure distributions for a gun was presented. The model used two coefficients and an exponent. The presence of the exponent greatly improved the polynomial model. These representations were then used in a functional using a lumped element model to determine the polynomial coefficients representing the velocity and pressure distributions. The functional provided a means to calculate the velocity and pressure distributions using only the configuration parameters of the gun and bullet and the powder's impetus value. No knowledge of a powder's burning rate or pressure index is required.

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